

# RECOVERING THE TWISTING FUNCTION IN A TWISTED WAVEGUIDE FROM THE DN MAP

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**ABSTRACT.** We consider the inverse problem of determining the twisting function in a infinite cylindrical twisted waveguide from the corresponding DN map. This problem, which is naturally linked to some inverse anisotropic conductivity problem in a straight waveguide, remains generally open, unless the twisting function is assumed to be affine. Namely we prove Lipschitz stability in the determination of affine twisting functions from the DN map. This result still holds true upon substituting a suitable approximation of the DN map, provided the first derivative of the twisting is sufficiently close to some *a priori* fixed constant.

**Key words :** Dirichlet Laplacian, twisted infinite cylindrical waveguide, twisting function, DN map, stability estimate.

**AMS subject classifications :** 35R30.

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## 1. INTRODUCTION

Let  $\omega$  be a bounded domain of  $\mathbb{R}^2$ . To  $\Omega = \omega \times \mathbb{R}$  and  $\theta \in C^1(\mathbb{R})$  we associate the infinite twisted cylindrical domain

$$\Omega_\theta = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \omega, x_3 \in \mathbb{R}\},$$

where  $R_\xi$  denotes the rotation in  $\mathbb{R}^2$  of angle  $\xi \in \mathbb{R}$ . Twisted waveguides modeled by  $\Omega_\theta$  exhibit interesting propagation properties such as the occurrence of propagating waveguide modes with phase velocities slower than those of similar modes in a straight waveguide. This explains why twisted waveguides are at the center of the attention of many theoretical and applied physicists (see e.g. [Ka, DR, KF, NZG, Sh, Wi, YM]). Moreover, it turns out that they are the source of challenging spectral and PDE problems, some of them having been extensively studied in the mathematical literature (see e.g. [BK, EKK, KK, KS, KZ1, KZ2]).

Nevertheless, the inverse problem of identifying the twisting function from the Dirichlet-to-Neumann (DN in short) map, has, to our knowledge, not been examined in this framework yet. The study of this open problem is actually the main purpose of the present article. Namely, we consider in this paper the following boundary value problem for the Laplacian in the twisted waveguide  $\Omega_\theta$ ,

$$(1.1) \quad \begin{cases} \Delta v(y) = 0, & y \in \Omega_\theta, \\ v(y) = g(y), & y \in \partial\Omega_\theta, \end{cases}$$

and we address the problem of recovering the twisting function  $\theta$  (actually its first derivative) from the DN map

$$(1.2) \quad \tilde{\Lambda}_\theta : g \rightarrow B(y) \nabla v(y) \cdot \nu(y),$$

where

$$B(y) = \begin{pmatrix} 1 & 0 & -y_2\theta'(y_3) \\ 0 & 1 & y_1\theta'(y_3) \\ 0 & 0 & 1 \end{pmatrix}.$$

More specifically, our aim is the stability issue for the problem of determining  $\theta$  from  $\tilde{\Lambda}_\theta$ .

The DN map  $\tilde{\Lambda}_\theta$  acts on functional spaces depending on  $\theta$ . Therefore it is not well suited to the analysis of this inverse problem. This difficulty may be overcome by turning (1.1)-(1.2) into an equivalent system associated to some  $\theta$ -independent DN map. To this purpose we introduce

$$T_\xi = \begin{pmatrix} R_\xi & 0 \\ 0 & 1 \end{pmatrix}$$

and put  $u(x) = v(T_{\theta(x_3)}(x', x_3))$ ,  $x = (x', x_3) \in \Omega$ . By performing the change of variable  $y = T_{\theta(x_3)}(x', x_3)$  in (1.1), we find by direct calculation that  $u$  is the solution to the following boundary value problem with an elliptic operator in the divergence form

$$(1.3) \quad \begin{cases} \operatorname{div}(A(x', \theta'(x_3))\nabla u) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases}$$

where,  $f(x) = g(T_{\theta(x_3)}(x', x_3))$ ,  $x \in \partial\Omega$ , and the matrix  $A$  is given by

$$A(x', t) = \begin{pmatrix} 1 + x_2^2 t^2 & -x_2 x_1 t^2 & -x_2 t \\ -x_2 x_1 t^2 & 1 + x_1^2 t^2 & x_1 t \\ -x_2 t & x_1 t & 1 \end{pmatrix}, \quad x' \in \omega, \quad t \in \mathbb{R}.$$

Moreover, it holds true that

$$(1.4) \quad B(y)\nabla v(y) \cdot \nu(y) = A(x', \theta'(x_3))\nabla u(x) \cdot \nu(x), \quad y = T_{\theta(x_3)}(x', x_3), \quad x = (x', x_3) \in \Omega.$$

This identity justifies the choice of the boundary operator  $B$  appearing in (1.2), which provides the appropriate Neumann condition, given by the right hand side of (1.4), on  $u$ . Formally, (1.4) indicates that recovering  $\theta$  from  $\tilde{\Lambda}_\theta$  is the same as determining  $\theta$  from the following DN map

$$\Lambda_\theta : f \rightarrow A\nabla u \cdot \nu.$$

The major part of our work will therefore be devoted to studying  $\Lambda_\theta$  in view of establishing stability in the identification of  $\theta$  from  $\Lambda_\theta$ . In light of (1.3)-(1.4) we notice that this is the same kind of inverse anisotropic conductivity problem, but stated here in an unbounded domain, as the one studied in a bounded domain by Alessandrini [A] and Alessandrini and Gaburro [AG1], [AG2] (see also Gaburro and Lionheart [GL]). However, it turns out that the usual monotonicity assumption on the conductivity, which is essential to the identification of  $A$  from the DN map in this approach, is not fulfilled by the matrix  $A$  under consideration. This is the main reason why the inverse problem associated to (1.3) remains open for general twisting functions  $\theta \in C^1(\mathbb{R})$ . Nevertheless, in this paper we are able to prove Lipschitz stability in the determination of  $\theta$  from the DN map when  $\theta$  is either an affine function or sufficiently close to an arbitrarily fixed constant.

The paper is organized as follows. Section 2 contains the definition and the main properties of  $\Lambda_\theta$ , needed in the proofs of the coming sections. The first part of section 3 explains why the approach developed by Alessandrini and Gaburro in [AG1] and [AG2] does not apply to the problem under consideration. Nevertheless, we prove in the second part of section 3 that twisting functions close enough to some arbitrarily fixed constant may actually be identified by solving an inverse conductivity problem which satisfies the above mentioned monotonicity condition. Further, the particular case of affine twisting functions is examined in section 4. This is by means of the Fourier transform with respect to the variable  $x_3$ , in order to bring the original problem into some anisotropic conductivity problem stated in  $\omega$ . The corresponding conductivity matrix satisfies a weak monotonicity condition, allowing to claim stability in the determination of the twisting function from some suitable DN map. Finally, the original DN map  $\tilde{\Lambda}_\theta$  is rigorously defined and linked to  $\Lambda_\theta$  in section 5.

## 2. THE DN MAP

**Extension lemma.** As we are dealing with an infinitely extended domain  $\Omega$ , we start by proving the following useful extension lemma.

**Lemma 2.1.** *Let  $g \in H^{s+1/2}(\mathbb{R}; H^s(\partial\omega))$  for  $s = 3/2$  or  $s = 1/2$ . Then there exists  $G \in H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega))$  such that  $G(t) = g(t)$  on  $\partial\omega$  and*

$$(2.1) \quad \|G\|_{H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega))} \leq C(\omega) \|g\|_{H^{s+1/2}(\mathbb{R}; H^s(\partial\omega))},$$

where  $C(\omega)$  is a constant depending only on  $\omega$ .

*Proof.* We detail the proof for  $s = 3/2$ , the case of  $s = 1/2$  being treated in a similar way. Let us first assume that  $g \in C_0^\infty(\mathbb{R}; H^{3/2}(\partial\omega))$ . For each  $h \in H^{3/2}(\partial\omega)$ , the boundary value problem

$$(2.2) \quad \begin{cases} \Delta H = 0 & \text{in } \omega, \\ H = h & \text{on } \partial\omega, \end{cases}$$

admits a unique solution  $H \in H^2(\omega)$ , according to [LM]. Moreover there exists a constant  $C(\omega)$ , depending only on  $\omega$ , such that the following estimate holds true:

$$(2.3) \quad \|H\|_{H^2(\omega)} \leq C(\omega) \|h\|_{H^{3/2}(\partial\omega)}.$$

Let  $G(t)$ ,  $t \in \mathbb{R}$ , be the solution of (2.2) corresponding to  $h = g(t)$ . Since  $G(t) - G(s)$ ,  $s, t \in \mathbb{R}$ , is the solution to the boundary value problem (2.2) with  $h = g(t) - g(s)$ , we deduce from (2.3) that

$$\|G(t) - G(s)\|_{H^2(\omega)} \leq C(\omega) \|g(t) - g(s)\|_{H^{3/2}(\partial\omega)}.$$

Therefore  $G \in C(\mathbb{R}; H^2(\omega))$ , and we have

$$(2.4) \quad \|G\|_{L^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g\|_{L^2(\mathbb{R}; H^{3/2}(\partial\omega))},$$

from (2.3). Let us next consider the solution  $K(t)$  to the boundary value problem (2.2) associated to  $h = g'(t)$ . Similarly we have  $K \in C(\mathbb{R}; H^2(\omega))$ , and (2.3) yields

$$\|G(t+s) - G(t) - sK(t)\|_{H^2(\omega)} \leq C(\omega) \|g(t+s) - g(t) - sg'(t)\|_{H^{3/2}(\partial\omega)},$$

since  $G(t+s) - G(t) - sK(t)$ ,  $t, s \in \mathbb{R}$ , is the solution of (2.2) associated to  $h = g(t+s) - g(t) - sg'(t)$ . From this then follows that  $G \in C^1(\mathbb{R}; H^2(\omega))$ ,  $G' = K$ , and

$$(2.5) \quad \|G'\|_{L^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g'\|_{L^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

By substituting  $G'$  for  $G$  in the above reasoning, we get that  $G \in C^2(\mathbb{R}; H^2(\omega))$ , and

$$(2.6) \quad \|G''\|_{L^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g''\|_{L^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Putting (2.4)-(2.6) together, we thus find out that

$$(2.7) \quad \|G\|_{H^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Further,  $g \in H^2(\mathbb{R}; H^{3/2}(\partial\omega))$  being fixed, we consider a sequence  $(g_n)_n$  in  $C_0^\infty(\mathbb{R}; H^{3/2}(\partial\omega))$  converging to  $g$  in  $H^2(\mathbb{R}; H^{3/2}(\partial\omega))$ . By uniqueness,  $G_n - G_m$ , where  $G_n$  (resp.  $G_m$ ) denotes the corresponding extension of  $g_n$  (resp.  $g_m$ ), extends  $g_n - g_m$  to  $H^2(\mathbb{R}; H^2(\omega))$ . Therefore, (2.7) yields

$$\|G_n - G_m\|_{H^2(\mathbb{R}; H^2(\omega))} \leq C(\omega) \|g_n - g_m\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))},$$

so  $(G_n)_n$  is a Cauchy sequence in  $H^2(\mathbb{R}; H^2(\omega))$ . We call  $G$  its limit. From the continuity of the trace operator

$$W \in H^2(\mathbb{R}; H^2(\omega)) \rightarrow W|_{\partial\Omega} \in H^2(\mathbb{R}; H^{3/2}(\partial\omega)),$$

we see that  $G$  extends  $g$ . Thus we end up getting (2.1) by applying (2.7) for  $g_n$ ,  $n \in \mathbb{N}$ , and sending  $n$  to infinity.  $\square$

Fix  $s = 3/2$  or  $s = 1/2$ . Since the trace operator  $w \in H^{s+1/2}(\omega) \rightarrow w|_{\partial\omega} \in H^s(\partial\omega)$  is bounded, then the same is true for

$$G \in H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega)) \rightarrow G|_{\partial\Omega} \in H^{s+1/2}(\mathbb{R}; H^s(\partial\omega)).$$

Hence,

$$\|g\|_{H^{s+1/2}(\mathbb{R}; H^s(\partial\omega))} = \inf\{\|G\|_{H^{s+1/2}(\mathbb{R}; H^{s+1/2}(\omega))}; G = g \text{ on } \partial\Omega\},$$

is a norm on  $H^{s+1/2}(\mathbb{R}; H^s(\partial\omega))$ , which is equivalent to the usual one. In the sequel, we shall use either one of these two equivalent norms, each of them being denoted by the same above mentioned symbol.

**Solution to the boundary value problem (1.3).** Let us first make the following remark on the uniform ellipticity of  $A$ , where  $A$  denotes either  $A(x', t)$  or  $A(x', \theta(x_3))$ , as defined in the previous section. For all  $\zeta \in \mathbb{R}^3$ ,  $x' \in \omega$  and  $t \in \mathbb{R}$ , we have

$$\begin{aligned} A(x', t)\zeta \cdot \zeta &= \zeta_1^2 + \zeta_2^2 + \zeta_3^2 - 2tx_2\zeta_1\zeta_3 + 2tx_1\zeta_2\zeta_3 + t^2(x_2\zeta_1 - x_1\zeta_2)^2 \\ &= \zeta_1^2 + \zeta_2^2 + (\zeta_3 + t(x_2\zeta_1 - x_1\zeta_2))^2, \quad x' = (x_1, x_2) \in \omega, t \in \mathbb{R}, \end{aligned}$$

by a straightforward computation. For every  $x' \in \omega$  and  $t \in \mathbb{R}$ , this entails that  $A(x', t)\zeta \cdot \zeta = 0$  if and only if  $\zeta = 0$ . Therefore, since  $\overline{\omega} \times [\underline{t}, \bar{t}]$  is compact for all real numbers  $\underline{t} < \bar{t}$ , there exists  $\lambda \geq 1$ , depending on  $\omega$ ,  $\underline{t}$  and  $\bar{t}$ , such that we have

$$(2.8) \quad \lambda^{-1}|\zeta|^2 \leq A(x', t)\zeta \cdot \zeta \leq \lambda|\zeta|^2 \text{ for all } x' \in \omega, t \in [\underline{t}, \bar{t}], \zeta \in \mathbb{R}^3.$$

In order to define the DN map associated to the boundary value problem (1.3), we first need to solve this later. To this end, pick  $f \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  and  $F \in H^1(\Omega)$  such that  $F = f$  on  $\partial\Omega$ , where  $H^1(\mathbb{R}, H^1(\omega))$  is identified with  $H^1(\Omega)$ . Notice that the existence of such a function  $F$  is guaranteed by Lemma 3.1. In light of (2.8) and the Lax-Milgram lemma, there is a unique  $v \in H_0^1(\Omega)$  solving the variational problem

$$(2.9) \quad \int_{\Omega} A \nabla v \cdot \nabla w dx = - \int_{\Omega} A \nabla F \cdot \nabla w dx, \text{ for all } w \in H_0^1(\Omega).$$

Hence  $u = v + F$  is the unique weak solution to the boundary value problem (1.3). That is,  $u$  satisfies the first equation in (1.3) in the distributional sense and the second equation in the trace sense. Moreover, taking  $w = v$  in (2.9), we obtain from Poincaré's inequality (which holds true for  $\Omega$  since  $\omega$  is bounded) that  $\|v\|_{H^1(\Omega)} \leq C\|F\|_{H^1(\Omega)}$  for some constant  $C > 0$  depending on  $\omega$ . Therefore we have

$$\|u\|_{H^1(\Omega)} \leq C\|F\|_{H^1(\Omega)},$$

where  $C$  denotes some generic positive constant depending on  $\omega$ . Finally, as  $F$  may be chosen in  $H^1(\Omega)$  so that  $F = f$  in  $\partial\Omega$ , we deduce from this and (2.1) that

$$(2.10) \quad \|u\|_{H^1(\Omega)} \leq C\|f\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))}.$$

**Definition of the DN map.** Prior to defining the DN map we need some technical result stated in the coming proposition. To this purpose we preliminarily introduce the following  $H(\text{div})$ -type space,

$$H(\text{div}_A, \Omega) = \{P \in L^2(\Omega)^3; \text{div}(AP) \in L^2(\Omega)\},$$

and recall that the dual space of  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  is denoted by  $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ .

**Proposition 2.1.** *Let  $P \in H(\text{div}_A, \Omega)$ . Then  $AP \cdot \nu \in H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$  and*

$$(2.11) \quad \|AP \cdot \nu\|_{H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))} \leq C(\|P\|_{L^2(\Omega)} + \|\text{div}(AP)\|_{L^2(\Omega)}).^1$$

*In addition, the following identity*

$$(2.12) \quad \langle AP \cdot \nu, g \rangle = \int_{\Omega} G \text{div}(AP) dx + \int_{\Omega} A \nabla G \cdot P dx,$$

*holds true for any  $g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  and  $G \in H^1(\Omega)$  such that  $G = g$  on  $\partial\Omega$ . Here  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  and its dual  $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ .*

<sup>1</sup>This means that the operator  $P \in C_0^\infty(\overline{\Omega}) \rightarrow AP \cdot \nu \in C^\infty(\partial\Omega)$  can be extended to a bounded operator from  $H(\text{div}_A, \Omega)$  into  $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ .

*Proof.* We first consider the case of  $P \in C_0^\infty(\overline{\Omega})^3$ . Fix  $g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  and let  $G \in H^1(\Omega)$  be chosen in accordance to Lemma 3.1 so that  $G = g$  on  $\partial\Omega$ . Since  $P$  has a compact support, we have

$$(2.13) \quad \int_{\Omega} G \operatorname{div}(AP) dx = - \int_{\Omega} A \nabla G \cdot P dx + \int_{\partial\Omega} g AP \cdot \nu d\sigma,$$

from Green's formula, whence

$$\left| \int_{\partial\Omega} g AP \cdot \nu d\sigma \right| \leq C \|G\|_{H^1(\Omega)} (\|P\|_{L^2(\Omega)} + \|\operatorname{div}(AP)\|_{L^2(\Omega)}).$$

By taking the infimum over  $\{G \in H^1(\Omega), G = g \text{ on } \partial\Omega\}$  in the right hand side of above estimate, we find that (2.11) holds true for every  $P \in C_0^\infty(\overline{\Omega})^3$ .

Further, pick  $P \in H(\operatorname{div}_A, \Omega)$ . The set  $C_0^\infty(\overline{\Omega})^3$  being dense in  $H(\operatorname{div}_A, \Omega)$ , as can be seen by mimicking the proof of Theorem 2.4 in [GR], we may find a sequence  $(P_k)_k$  in  $C_0^\infty(\overline{\Omega})^3$  converging to  $P$  in  $H(\operatorname{div}_A, \Omega)$ . Moreover, due to (2.11),  $(AP_k \cdot \nu)_k$  is a Cauchy sequence in  $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ . Therefore  $(AP_k \cdot \nu)_k$  has a limit in  $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ , which is denoted by  $AP \cdot \nu$ , and (2.12) follows readily from (2.13).  $\square$

Let  $u$  denote the  $H^1(\Omega)$ -solution to (1.3). By applying Proposition 2.1 to  $P = \nabla u$ , we deduce from (2.10) that

$$\Lambda_\theta : f \rightarrow A \nabla u \cdot \nu$$

is well defined as a bounded operator from  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  into  $H^{-1}(\mathbb{R}; H^{-1/2}(\partial\omega))$ . Moreover the following identity

$$(2.14) \quad \langle \Lambda_\theta f, g \rangle = \int_{\Omega} A \nabla u \cdot \nabla G dx,$$

holds true for all  $g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  and  $G \in H^1(\Omega)$  such that  $G = g$  on  $\partial\Omega$ .

Further, by taking  $G = v$  in (2.14), where  $v$  is the solution to (1.3) with  $f$  replaced by  $g$ , we find out that

$$\langle \Lambda_\theta f, g \rangle = \int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} \nabla u \cdot A \nabla v dx.$$

Therefore we have

$$(2.15) \quad \langle \Lambda_\theta f, g \rangle = \langle f, \Lambda_\theta g \rangle, \text{ for all } f, g \in H^1(\mathbb{R}; H^{1/2}(\partial\omega)).$$

This proves that  $\Lambda_\theta^*|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} = \Lambda_\theta$ , where  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  is identified with a subspace of its bidual space.

Finally, for  $i = 1, 2$ , put  $A_i = A(x', \theta_i(x_3))$  and  $\Lambda_i = \Lambda_{\theta_i}$ , and let  $u_i \in H^1(\Omega)$ ,  $i = 1, 2$ , be a weak solution to the equation

$$\operatorname{div}(A_i \nabla u_i) = 0 \text{ in } \Omega.$$

By applying (2.14) to  $f = u_i|_{\partial\Omega}$  and  $g = u_{3-i}|_{\partial\Omega}$ ,  $i = 1, 2$ , we get that

$$\langle \Lambda_1 u_1, u_2 \rangle = \int_{\Omega} A_1 \nabla u_1 \cdot \nabla u_2 dx \text{ and } \langle \Lambda_2 u_2, u_1 \rangle = \int_{\Omega} A_2 \nabla u_2 \cdot \nabla u_1 dx.$$

In light of (2.15), this yields:

$$(2.16) \quad \langle (\Lambda_1 - \Lambda_2) u_1, u_2 \rangle = \int_{\Omega} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx.$$

**Restriction of the DN map.** We turn now to establishing some smoothness property for the restriction of  $\Lambda_\theta$  to  $H^2(\mathbb{R}; H^{3/2}(\partial\omega))$ . We shall assume for this purpose that  $\Omega_1 = \omega \times (-1, 1)$  has  $H^2$ -regularity property, i.e. that for every  $f \in L^2(\Omega)$  and any matrix-valued function  $C = (C_{ij}(x))_{1 \leq i, j \leq 3}$  with coefficients in  $W^{1, \infty}(\Omega_1)$  satisfying the ellipticity condition

$$\exists \alpha > 0, \quad C(x) \xi \cdot \xi \geq \alpha |\zeta|^2, \text{ for all } \zeta \in \mathbb{C}^3, \quad x \in \Omega_1,$$

the following boundary value problem

$$\begin{cases} \operatorname{div}(C\nabla w) = f & \text{in } \Omega_1, \\ w = 0 & \text{on } \partial\Omega_1, \end{cases}$$

has a unique solution  $w \in H^2(\Omega_1)$  obeying

$$\|w\|_{H^2(\Omega_1)} \leq C(\alpha, M)\|f\|_{L^2(\Omega_1)},$$

for some constant  $C(\alpha, M) > 0$  depending only on  $\alpha$ ,  $M = \max_{1 \leq i, j \leq 3} \|C_{ij}\|_{W^{1,\infty}(\Omega_1)}$  and  $\omega$ .

We notice that  $\Omega_1$  has  $H^2$ -regularity property if and only if  $\Omega_a = \omega \times (-a, a)$  has  $H^2$ -regularity property for some  $a > 0$ , and from [Gr] that  $\Omega_1$  has  $H^2$ -regularity property provided  $\omega$  is convex.

Having said that we may now prove, upon identifying  $H^2(\Omega)$  with  $H^2(\mathbb{R}; H^2(\omega))$ , the following claim, which is a cornerstone in the derivation of smoothness properties for the restriction of  $\Lambda_\theta$  to  $H^2(\mathbb{R}; H^{3/2}(\partial\omega))$ .

**Theorem 2.1.** *Assume that  $\theta \in C^{1,1}(\mathbb{R})$  and that  $\Omega_1$  has  $H^2$ -regularity property. Then for any  $f \in H^2(\mathbb{R}; H^{3/2}(\partial\omega))$ , the boundary value problem (1.3) has a unique solution  $u \in H^2(\Omega)$ . Moreover if  $\|\theta\|_{C^{1,1}(\mathbb{R})} \leq M$ , for some  $M > 0$ , we may find a constant  $C > 0$ , depending only on  $M$  and  $\omega$ , such that we have:*

$$(2.17) \quad \|u\|_{H^2(\Omega)} \leq C\|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

*Proof.* For  $M > 0$  fixed, let  $\theta \in C^{1,1}(\mathbb{R})$  be such that  $\|\theta\|_{C^{1,1}(\mathbb{R})} \leq M$  and pick  $f \in H^2(\mathbb{R}; H^{3/2}(\partial\omega))$ . We know from Lemma 3.1 that there exists  $F \in H^2(\mathbb{R}; H^{3/2}(\omega))$  such that  $F = f$  on  $\partial\Omega$  and

$$(2.18) \quad \|F\|_{H^2(\Omega)} \leq C(\omega)\|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

Set

$$\Psi = \operatorname{div}(A\nabla F).$$

In light of (2.8) there is a unique  $u_0 \in H_0^1(\Omega)$  satisfying simultaneously

$$(2.19) \quad \int_{\Omega} A\nabla u_0 \cdot \nabla v dx = \int_{\Omega} \Psi v dx, \text{ for all } v \in H_0^1(\Omega),$$

and

$$(2.20) \quad \|u_0\|_{H^1(\Omega)} \leq C_0\|\Psi\|_{L^2(\Omega)},$$

for some constant  $C_0 > 0$  depending on  $\omega$  and  $M$ .

Further, we consider  $\xi_n \in C_0^\infty(-(n+1), n+1)$ , for  $n \geq 1$ , such that  $\xi_n = 1$  in  $[-n, n]$  and  $\|\xi'_n\|_\infty \leq 1/2$ ,  $\|\xi''_n\|_\infty \leq 1/2$ . For every  $v \in H_0^1(\Omega)$ , we get using standard computations that

$$\int_{\Omega} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega} A\nabla u_0 \cdot \nabla(\xi_n v) dx - \int_{\Omega} (A\nabla u_0 \cdot \nabla \xi_n) v dx + \int_{\Omega} (A\nabla \xi_n \cdot \nabla v) u_0 dx.$$

An integration by parts in the last term of the right hand side of the above identity providing

$$\int_{\Omega} (A\nabla \xi_n \cdot \nabla v) u_0 dx = - \int_{\Omega} (A\nabla \xi_n \cdot \nabla u_0) v dx - \int_{\Omega} \operatorname{div}(A\nabla \xi_n) u_0 v dx,$$

we next find out that

$$\int_{\Omega} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega} A\nabla u_0 \cdot \nabla(\xi_n v) dx - \int_{\Omega} (A\nabla u_0 \cdot \nabla \xi_n) v dx - \int_{\Omega} (A\nabla \xi_n \cdot \nabla u_0) v dx - \int_{\Omega} \operatorname{div}(A\nabla \xi_n) u_0 v dx.$$

Since  $A$  is symmetric, it follows from the above equality (2.19) that

$$\int_{\Omega} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega} \Psi \xi_n v dx - 2 \int_{\Omega} (A\nabla \xi_n \cdot \nabla u_0) v dx - \int_{\Omega} \operatorname{div}(A\nabla \xi_n) u_0 v dx, \text{ for all } v \in H_0^1(\Omega).$$

As a consequence the function  $\xi_n u_0 \in H_0^1(\Omega_{n+1})$ , where we recall that  $\Omega_a = \omega \times (-a, a)$  for any  $a > 0$ , is the solution to the variational problem

$$(2.21) \quad \int_{\Omega_{n+1}} A\nabla(\xi_n u_0) \cdot \nabla v dx = \int_{\Omega_{n+1}} \tilde{\Psi} v dx, \text{ for all } v \in H_0^1(\Omega_{n+1}),$$

with

$$\tilde{\Psi} = \Psi \xi_n - 2A \nabla \xi_n \cdot \nabla u_0 - \operatorname{div}(A \nabla \xi_n) u_0.$$

The next step of the proof is to make the change of variables  $(x', x_3) \in \Omega_{n+1} \rightarrow (x', y_3) = (x', 1/(n+1)x_3) \in \Omega_1$  in (2.21). To this end, we set

$$J_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{pmatrix}, \quad n \geq 1,$$

and for all  $(x', y_3) \in \Omega_1$ , we introduce the following notations:

$$\begin{aligned} \underline{A}(x', y_3) &= 1/(n+1) J_{n+1} A(x', (n+1)y_3) J_{n+1}, \\ \underline{\xi}(y_3) &= \xi_n((n+1)y_3), \\ \underline{u}(x', y_3) &= u_0(x', (n+1)y_3), \\ w_n(x', y_3) &= \xi_n((n+1)y_3) u_0(x', (n+1)y_3), \\ \underline{\operatorname{div}}(P(x', y_3)) &= \partial_{x_1} P_1(x', y_3) + \partial_{x_2} P_2(x', y_3) + 1/(n+1) \partial_{y_3} P_3(x', y_3), \\ \underline{\Psi}(x', y_3) &= 1/(n+1) \left[ \Psi(x', (n+1)y_3) - 2J_{n+1} A(x', (n+1)y_3) J_{n+1} \nabla \underline{\xi}(y_3) \cdot \nabla \underline{u}(x', y_3) \right. \\ &\quad \left. - \underline{\operatorname{div}}(A(x', (n+1)y_3) J_{n+1} \nabla \underline{\xi}(y_3)) \underline{u}(x', y_3) \right]. \end{aligned}$$

By performing the above mentioned change of variables, we find out that  $w_n \in H_0^1(\Omega_1)$  is the solution of the variational problem

$$\int_{\Omega_1} \underline{A} \nabla w_n \cdot \nabla v dx = \int_{\Omega_1} \underline{\Psi} v dx, \quad \text{for all } v \in H_0^1(\Omega_1).$$

As  $\Omega_1$  has  $H^2$ -regularity property by assumption, we get by straightforward computations that  $\|w_n\|_{H^2(\Omega_1)} \leq C(M, \omega) \|\underline{\Psi}\|_{L^2(\Omega_1)}$ . On the other hand, since  $\|\underline{\Psi}\|_{L^2(\Omega_1)} \leq (n+1)^{-3/2} C(M, \omega) \|\Psi\|_{L^2(\Omega)}$ , from (2.20), it holds true that

$$(2.22) \quad \|w_n\|_{H^2(\Omega_1)} \leq (n+1)^{-3/2} C(M, \omega) \|\Psi\|_{L^2(\Omega)}.$$

Moreover, putting (2.20) and (2.22) together we obtain that

$$(2.23) \quad \|\xi_n u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|\Psi\|_{L^2(\Omega)}.$$

Therefore, upon eventually extracting a subsequence, we may assume that  $(\xi_n u_0)_n$  converges weakly to  $\tilde{u}$  in  $H^2(\Omega)$ . On the other hand, since  $(\xi_n u_0)_n$  converges to  $u_0$  in  $L^2(\Omega)$ , we have  $u_0 \in H^2(\Omega)$  and thus  $(\xi_n u_0)_n$  converges weakly to  $u_0$  in  $H^2(\Omega)$ . Further, the norm  $\|\cdot\|_{H^2(\Omega)}$  being lower semi-continuous, we have

$$\|u_0\|_{H^2(\Omega)} \leq \liminf_n \|\xi_n u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|\Psi\|_{L^2(\Omega)},$$

from (2.23). Bearing in mind that  $\|\Psi\|_{L^2(\Omega)} \leq C(M, \omega) \|F\|_{H^2(\Omega)}$  and  $\|F\|_{H^2(\Omega)} \leq C(\omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}$ , this entails that

$$\|u_0\|_{H^2(\Omega)} \leq C(M, \omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

As a consequence  $u = u_0 + F \in H^2(\Omega)$  is the unique solution to (1.3), and it satisfies in addition

$$\|u\|_{H^2(\Omega)} \leq C(M, \omega) \|f\|_{H^2(\mathbb{R}; H^{3/2}(\partial\omega))}.$$

This proves the result.  $\square$

For each  $\theta$  obeying the assumptions of Theorem 2.1, the mapping

$$\Lambda_\theta : f \in H^2(\mathbb{R}; H^{3/2}(\partial\omega)) \rightarrow \partial_\nu u \in H^2(\mathbb{R}; H^{1/2}(\partial\omega)),$$

where  $u$  denotes the unique  $H^2(\Omega)$ -solution to (1.3), is well defined by Theorem 2.1.

Further, bearing in mind that  $C_0^\infty(\mathbb{R}; H^2(\omega))$  is dense in  $H^2(\Omega)$  and arguing in the same way as in the derivation of Lemma 3.1, we find out that the trace operator

$$\tilde{\tau} : w \in H^2(\Omega) \rightarrow \partial_\nu w \in H^2(\mathbb{R}; H^{1/2}(\partial\omega))$$



is bounded. Moreover, we have  $\|\tilde{\tau}\| \leq \|\tau\|$ , where  $\tau : w \in H^2(\omega) \rightarrow \partial_\nu w \in H^{1/2}(\partial\omega)$  denotes the usual trace operator. From this and (2.17) then follows that  $\|\Lambda_\theta\| \leq C(M, \omega)$ , as a linear bounded operator from  $H^2(\mathbb{R}; H^{3/2}(\partial\omega))$  into  $H^2(\mathbb{R}; H^{1/2}(\partial\omega))$ .

### 3. DETERMINING THE TWISTING FUNCTION FROM THE DN MAP: AN OPEN PROBLEM

**Open problem.** We first detail the reason why the problem of the identification of  $\theta$  from  $\Lambda_\theta$  remains open, at least in its full generality. To this purpose we consider a nonempty open subset  $\gamma$  of  $\partial\omega$  and  $\Gamma = \gamma \times (-2L, 2L)$ , for some fixed  $L > 0$ . Next we introduce the two following functional spaces

$$H_\Gamma^1(\mathbb{R}; H^{1/2}(\partial\omega)) = \{f \in H^1(\mathbb{R}; H^{1/2}(\partial\omega)); \text{supp } f \subset \Gamma\},$$

and

$$\mathcal{W}(\mathbb{R}) = \{\theta \in W_{loc}^{2,\infty}(\mathbb{R}); \theta' \in W^{1,\infty}(\mathbb{R})\}^2.$$

Further, for  $i = 1, 2$ , we pick  $\theta_i \in \mathcal{W}(\mathbb{R})$  and set  $A_i = A(x', \theta'_i(x_3))$ ,  $\Lambda_i = \Lambda_{\theta_i}$ . In light of (2.15), we see that

$$(3.1) \quad \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle = \int_\Omega (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx,$$

for any function  $u_i \in H^1(\Omega)$ ,  $i = 1, 2$ , which is a weak solution to the equation  $\text{div}(A_i \nabla u_i) = 0$  in  $\Omega$ . Assume that

$$\theta_1(x_3) = \theta_2(x_3), \quad |x_3| > L.$$

Then we deduce from (3.1) that

$$(3.2) \quad \langle (\Lambda_1 - \Lambda_2)u_1, u_2 \rangle = \int_{\Omega^L} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx,$$

where  $\Omega^L = \omega \times (-L, L)$ .

Next, for  $0 < \rho \leq \rho_0$ , where  $\rho_0$  is some characteristic constant defined in [AG2] which depends only on  $\omega$  and  $L$ , we put

$$\begin{aligned} \Gamma_\rho &= \{x \in \Gamma; \text{dist}(x, \partial\Gamma) > \rho\}, \\ U_\rho &= \{x \in \mathbb{R}^3; \text{dist}(x, \Gamma_\rho) < \rho/4\}. \end{aligned}$$

Upon eventually shortening  $\rho_0$ , we may assume without loss of generality that  $\Gamma_0 = \gamma_0 \times [-L, L] \subset \Gamma_\rho$  for some  $\gamma_0 \Subset \gamma$ . Then, in view of [AG2], we can find a Lipschitz domain  $\Omega_\rho$  satisfying simultaneously:

$$\Omega \subset \Omega_\rho, \quad \Gamma_0 \subset \partial\Omega \cap \Omega_\rho \Subset \Gamma \text{ and } \text{dist}(x, \partial\Omega_\rho) \geq \rho/2 \text{ for all } x \in U_\rho.$$

Moreover we know from [AG1][Section 3] that there exists a unitary  $C^\infty$  vector field  $\tilde{\nu}$ , defined in some suitable neighborhood of  $\partial\omega \times (-2L, 2L)$ , which is non tangential to  $\partial\Omega$  and points to the exterior of  $\Omega$ . For  $x^0 \in \overline{\Gamma_\rho}$ , the point  $z_\tau = x^0 + \tau\tilde{\nu}$  obeys  $C\tau \leq \text{dist}(z_\tau, \partial\Omega) \leq \tau$  for all  $0 < \tau \leq \tau_0$ , according to [AG1][Lemma 3.3], where  $C$  and  $\tau_0$  are two positive constants depending only on  $\Omega$ ,  $\lambda$ ,  $\underline{t}$  and  $\bar{t}$ .

In light of [HK], the operator  $\text{div}(A_i \nabla \cdot)$ ,  $i = 1, 2$ , has a Dirichlet Green function  $G_i = G_i(x, y)$  on  $\Omega_\rho$ . More specifically  $G_i(x) = G_i(x, z_\tau)$ ,  $i = 1, 2$ , is the solution to the boundary value problem:

$$(3.3) \quad \begin{cases} \text{div}(A_i \nabla G_i) = -\delta(x - z_\tau) & \text{in } \mathcal{D}'(\Omega_\rho), \\ G_i = 0 & \text{on } \partial\Omega_\rho. \end{cases}$$

Moreover, the claim of [AG2][Corollary 3.4] remains essentially unchanged for the unbounded domain  $\Omega_\rho$  arising in this framework:

**Lemma 3.1.** *There exist two constants  $\tau_0 = \tau_0(\Omega, \rho) > 0$  and  $C = C(\Omega, \lambda, \underline{t}, \bar{t})$  such that the restriction  $G_i(\cdot, z_\tau)$  to  $\Omega$ , belongs to  $H^1(\Omega)$ , and satisfies*

$$(3.4) \quad \|G_i(\cdot, z_\tau)\|_{H^1(\Omega)} \leq C\tau^{-1/2}, \quad 0 < \tau \leq \tau_0.$$

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<sup>2</sup>We could choose  $\mathcal{W}(\mathbb{R}) = \{\theta \in \mathcal{D}'(\mathbb{R}); \theta' \in W^{1,\infty}(\mathbb{R})\}$  as well.



*Proof.* Choose  $\tau_0 = \tau_0(\Omega, \rho)$  so small relative to  $\rho$  that  $\text{dist}(z_\tau, \partial\Omega_\rho) > \tau$  for all  $\tau \in (0, \tau_0]$ . Since the ball centered at  $z_\tau$  with radius  $\tau/3$ , noted  $B_{\tau/3}(z_\tau)$ , is embedded in  $\Omega_\rho \setminus \overline{\Omega}$ , we have

$$(3.5) \quad \|G_i(\cdot, z_\tau)\|_{Y^{2,1}(\Omega)} \leq \|G_i(\cdot, z_\tau)\|_{Y^{2,1}(\Omega_\rho \setminus B_{\tau/3}(z_\tau))} \leq C\tau^{-1/2}, \quad 0 < \tau \leq \tau_0,$$

directly from [HK][Formula (4.45)]. Here  $C$  is some positive constant depending only on  $\Omega$ ,  $\lambda$ ,  $\underline{t}$  and  $\bar{t}$ , and the space

$$Y^{2,1}(\Omega) = \{u \in L^6(\Omega); \nabla u \in L^2(\Omega)^3\},$$

is endowed with the norm  $\|v\|_{Y^{2,1}(\Omega)} = \|v\|_{L^6(\Omega)} + \|\nabla v\|_{L^2(\Omega)^3}$ .

Finally,  $\Omega$  having the Poincaré inequality property since  $\omega$  is bounded, we may actually deduce (3.4) from (3.5).  $\square$

Now, as a Green function is a Levi function, it behaves locally like a parametrix. Hence, by applying [M][Formula (8.4)],  $G_i$ ,  $i = 1, 2$ , can be brought into the form

$$(3.6) \quad G_i(x) = C(\det(A_i(z_\tau)))^{-1/2} (A(z_\tau)^{-1}(x - z_\tau) \cdot (x - z_\tau))^{-1/2} + R_i(x),$$

where  $C > 0$  is a constant and the reminder  $R_i$  obeys the condition:

$$\exists(r_0, \alpha) \in \mathbb{R}_+^* \times (0, 1), \quad |R_i(x)| + |x - z_\tau| |\nabla R_i| \leq C|x - z_\tau|^{-1+\alpha}, \quad x \in \Omega_\rho, |x - z_\tau| \leq r_0.$$

Since

$$\text{div}(A_i \nabla G_i) = 0 \quad \text{in } \Omega,$$

in the weak sense, we can apply (3.2) ; whence

$$\langle (\Lambda_1 - \Lambda_2)G_1, G_2 \rangle = \int_{\Omega^L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx.$$

Moreover, taking into account that  $G_i|_{\partial\Omega} \in H_\Gamma^1(\mathbb{R}; H^{1/2}(\partial\omega))$ , we obtain that

$$\int_{\Omega^L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx \leq \|\Lambda_1^\Gamma - \Lambda_2^\Gamma\| \|G_1\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \|G_2\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))},$$

where  $\Lambda_i^\Gamma$ ,  $i = 1, 2$ , is the restriction of  $\Lambda_i$  to the closed subspace  $H_\Gamma^1(\mathbb{R}; H^{1/2}(\partial\omega))$ . As a consequence we have

$$(3.7) \quad \int_{\Omega^L} (A_1 - A_2) \nabla G_1 \cdot \nabla G_2 dx \leq C \|\Lambda_1^\Gamma - \Lambda_2^\Gamma\| \|G_1\|_{H^1(\Omega)} \|G_2\|_{H^1(\Omega)}.$$

Now, pick  $x^0 \in \Gamma_0$  such that  $|\theta'_1(x_3^0) - \theta'_2(x_3^0)| = \|\theta'_1 - \theta'_2\|_{L^\infty(-L, L)}$ . Actually, we may assume without loss of generality that we have  $|\theta'_1(x_3^0) - \theta'_2(x_3^0)| = \theta'_1(x_3^0) - \theta'_2(x_3^0)$ . In view of (3.6) and [AG2][Formula (4.2)], the main term in the left hand side of (3.7) has the following expression

$$\int_{B(z_\tau, \rho) \cap \Omega} \frac{[A^{-1}((x^0)', t_0) - A^{-1}((x^0)', s_0)](x - z_\tau) \cdot (x - z_\tau)}{[P_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2} [Q_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2}} dx,$$

with  $t_0 = \theta'_1(x_3^0)$ ,  $s_0 = \theta'_2(x_3^0)$ ,  $P_0 = A^{-1}(z'_\tau, t_0)$  and  $Q_0 = A^{-1}(z'_\tau, s_0)$ ,  $z_\tau$  being the same as in above, i.e.  $z_\tau = x^0 + \tau \tilde{\nu}$  for some  $\tau \in \mathbb{R}_+^*$ . The main ingredient in the analysis developped in [AG2] is the ellipticity condition [AG2][Formula (2.5)] imposed on  $\partial_t A(x', t)$ . Indeed, this assumption entails

$$(3.8) \quad \int_{B(z_\tau, \rho) \cap \Omega} \frac{[A^{-1}((x^0)', t_0) - A^{-1}((x^0)', s_0)](x - z_\tau) \cdot (x - z_\tau)}{[P_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2} [Q_0(x - z_\tau) \cdot (x - z_\tau)]^{3/2}} dx \geq C\tau(t_0 - s_0),$$

for some constant  $C > 0$ , which leads ultimately to the desired result. Unfortunately, in this framework, the ellipticity condition [AG2][Formula (2.5)] is not fulfilled by  $\partial_t A(x', t)$ . This can be seen from the following

explicit expression

$$\begin{aligned}\lambda_1 &= 0, \\ \lambda_2 &= |x'|^2 t - \sqrt{|x'|^4 t^2 + |x'|^2}, \\ \lambda_3 &= |x'|^2 t + \sqrt{|x'|^4 t^2 + |x'|^2},\end{aligned}$$

of the eigenvalues of  $\partial_t A(x', t)$ , showing that the spectrum of  $\partial_t A(x', t)$  has a negative component for  $x' \in \partial\omega$ . Moreover, it can be noticed that, due to the occurrence of this negative eigenvalue, the weak monotonicity assumption [AG1][Formula (5.7)] is not satisfied by the conductivity matrix under consideration either. Therefore, the approach developed in [AG2] does not apply to the problem under study. This explains why the determination of the twisting function from the corresponding DN map remains an open problem in the general case.

Nevertheless, we shall establish in the coming section that this is not the case for affine twisting functions anymore. Prior to examining this very peculiar framework, we first address the case of twisting functions which are close to some *a priori* fixed constant value.

**The case of twisting functions close to a constant value.** Put

$$A^*(x', t) = t \begin{pmatrix} 1 + x_2^2 & -x_2 x_1 & -x_2 \\ -x_2 x_1 & 1 + x_1^2 & x_1 \\ -x_2 & x_1 & 1 \end{pmatrix}, \quad x' \in \omega, \quad t \in \mathbb{R},$$

and denote by  $\Lambda_\theta^*$  the DN map  $\Lambda_\theta$  where  $A^*(x', \theta(x_3))$  is substituted for  $A(x', \theta(x_3))$ . Then, by arguing as in the derivation of (2.16), we obtain that

$$(3.9) \quad \langle (\Lambda_\theta - \Lambda_\theta^*)u, u^* \rangle = \int_{\Omega} (A(x', \theta(x_3)) - A^*(x', \theta(x_3))) \nabla u \cdot \nabla u^* dx,$$

for all solutions  $u, u^* \in H^1(\Omega)$  to the equations

$$\begin{aligned}\operatorname{div}(A(x', \theta(x_3)) \nabla u) &= 0 \quad \text{in } \Omega, \\ \operatorname{div}(A^*(x', \theta(x_3)) \nabla u^*) &= 0 \quad \text{in } \Omega,\end{aligned}$$

in the weak sense. Let us now assume that  $\|\theta' - 1\|_\infty \leq \frac{1}{2}^3$ . In light of (3.9), we get that

$$\|\Lambda_\theta - \Lambda_\theta^*\|_{\mathcal{L}(H^{-1}(\mathbb{R}, H^{-1/2}(\partial\omega)), H^1(\mathbb{R}, H^{1/2}(\partial\omega)))} \leq C \|\theta' - 1\|_\infty,$$

where  $C = C(\omega)$  is some positive constant. Therefore, if  $\theta'$  is sufficiently close to 1, the operator  $\Lambda_\theta^*$  may be seen as a suitable approximation of  $\Lambda_\theta$  in the inverse problem of determining  $\theta'$  from  $\Lambda_\theta$ . The main benefit of dealing with  $\Lambda_\theta^*$  instead of  $\Lambda_\theta$ , boils down to the fact that

$$\partial_t A^*(x', t) = A^*(x', 1) = A(x', 1), \quad \text{for all } x' \in \omega \text{ and } t \in \mathbb{R},$$

so the ellipticity condition [AG2][Formula (2.5)] required by the method developed in [AG2], is fulfilled by  $A^*$ . Therefore, by repeating the arguments of the proof of [AG2][Theorem 2.2], we obtain the:

**Theorem 3.1.** *Let  $L > 0$  and  $M > 0$ . Assume that  $\theta_1, \theta_2 \in \mathcal{W}(\mathbb{R})$  obey  $\theta'_1(x_3) = \theta'_2(x_3)$  for  $|x_3| > L$  and that*

$$\|\theta'_1\|_{W^{1,\infty}(\mathbb{R})}, \|\theta'_2\|_{W^{1,\infty}(\mathbb{R})} \leq M.$$

*Then there exists a constant  $C = C(M, \omega, L) > 0$  such that we have*

$$\|\theta'_1 - \theta'_2\|_{L^\infty(\mathbb{R})} \leq C \|(\Lambda_{\theta_1}^*)^\Gamma - (\Lambda_{\theta_2}^*)^\Gamma\|_{\mathcal{L}(H^{-1}(\mathbb{R}, H^{-1/2}(\partial\omega)), H^1(\mathbb{R}, H^{1/2}(\partial\omega)))}.$$

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<sup>3</sup>Note that 1 can be replaced by any constant.

## 4. THE CASE OF AFFINE TWISTING FUNCTIONS

In this section we address the case of affine twisting functions by means of the partial Fourier transform  $\mathcal{F}_{x_3}$  with respect to the variable  $x_3$ . This is suggested by the translational invariance of the system under consideration in the infinite direction  $x_3$ , arising from the fact that the matrix  $A(x', \theta'(x_3))$  appearing in (1.3) is independent of  $x_3$  in this peculiar case.

In the sequel, we note  $\xi$  the Fourier variable associated to  $x_3$  and we write  $\widehat{w}$  instead of  $\mathcal{F}_{x_3} w$  for every function  $w = w(x', x_3)$ :

$$\widehat{w}(x', \xi) = (\mathcal{F}_{x_3} w)(x', \xi), \quad x' \in \omega, \quad \xi \in \mathbb{R}.$$

The first step of the method is to re-express the system (1.3) in the Fourier plane  $\{(x', \xi), \quad x' \in \omega, \xi \in \mathbb{R}\}$ .

**Rewriting the boundary value problem in the Fourier domain.** We start with the two following useful technical lemmas.

**Lemma 4.1.** *For every  $w \in H^1(\Omega)$  it holds true that  $\widehat{\partial_{x_j} w} = \partial_{x_j} \widehat{w}$ ,  $j = 1, 2$ .*

*Proof.* Fix  $j = 1, 2$ . For every  $\varphi \in C_0^\infty(\omega)$  and  $\psi \in \mathcal{S}(\mathbb{R})$ , we have

$$\int_{\omega} \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' = \int_{\mathbb{R}} \widehat{\psi}(x_3) \left( \int_{\omega} \partial_{x_j} w(x', x_3) \varphi(x') dx' \right) dx_3,$$

from Fubini's theorem. By integrating by parts in the last integral, we obtain

$$\int_{\omega} \partial_{x_j} w(x', x_3) \varphi(x') dx' = - \int_{\omega} w(x', x_3) \partial_{x_j} \varphi(x') dx', \quad \text{a.e. } x_3 \in \mathbb{R},$$

so we get

$$\begin{aligned} \int_{\omega} \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' &= - \int_{\mathbb{R}} \widehat{\psi}(x_3) \left( \int_{\omega} w(x', x_3) \partial_{x_j} \varphi(x') dx' \right) dx_3 \\ &= - \int_{\omega} \partial_{x_j} \varphi(x') \left( \int_{\mathbb{R}} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx'. \end{aligned}$$

Further, the operator  $\mathcal{F}_{x_3}$  being selfadjoint in  $L^2(\mathbb{R})$ , it holds true that

$$\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \widehat{w}(x', \xi) \psi(\xi) d\xi, \quad \text{a.e. } x' \in \omega,$$

whence

$$\begin{aligned} \int_{\omega} \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' &= - \int_{\omega} \partial_{x_j} \varphi(x') \left( \int_{\mathbb{R}} \widehat{w}(x', \xi) \psi(\xi) d\xi \right) dx' \\ &= \int_{\omega} \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_j} \widehat{w}(x', \xi) \psi(\xi) d\xi \right) dx', \end{aligned}$$

by integrating by parts. From the density of  $C_0^\infty(\omega)$  in  $L^2(\omega)$ , the above identity entails that

$$\int_{\mathbb{R}} \partial_{x_j} w(x', x_3) \widehat{\psi}(x_3) dx_3 = \int_{\mathbb{R}} \partial_{x_j} \widehat{w}(x', \xi) \psi(\xi) d\xi, \quad \text{a.e. } x' \in \omega,$$

for every  $\psi \in \mathcal{S}(\mathbb{R})$ . From this, the selfadjointness of  $\mathcal{F}_{x_3}$  and the density of  $\mathcal{S}(\mathbb{R})$  in  $L^2(\mathbb{R})$ , then follows that  $\widehat{\partial_{x_j} w} = \partial_{x_j} \widehat{w}$ .  $\square$

**Lemma 4.2.** *Let  $C = (C_{kl})_{1 \leq k, l \leq 3} \in W^{1, \infty}(\omega)^{3 \times 3}$  be such that  $(C_{kl}(x'))_{1 \leq k, l \leq 3}$  is symmetric for any  $x' \in \omega$ . Then every  $w \in H^1(\Omega)$  obeying*

$$(4.1) \quad \int_{\Omega} C \nabla w \cdot \nabla v dx = 0 \quad \text{for all } v \in H_0^1(\Omega),$$

*satisfies the equation*

$$(4.2) \quad -\operatorname{div}_{x'}(\widetilde{C}(x') \nabla_{x'} \widehat{w}) + P(x', \xi) \cdot \nabla_{x'} \widehat{w} + q(x', \xi) \widehat{w} = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

where

$$\begin{aligned}\tilde{C}(x') &= (C_{ij}(x'))_{1 \leq i, j \leq 2} \\ P(x', \xi) &= -i2\xi \begin{pmatrix} C_{31}(x') \\ C_{32}(x') \end{pmatrix} \\ q(x', \xi) &= -i\xi \operatorname{div}_{x'} \begin{pmatrix} C_{31}(x') \\ C_{32}(x') \end{pmatrix} + \xi^2 C_{33}(x').\end{aligned}$$

Moreover, if  $w \in H^2(\Omega)$  is solution to (4.1) then the identity (4.2) holds true for a.e.  $(x', \xi) \in \Omega$ .

*Proof.* Choose  $v = \varphi \otimes \widehat{\psi}$  in (4.1), where  $\varphi \in C_0^\infty(\omega)$  and  $\psi \in \mathcal{S}(\mathbb{R})$ , so we have:

$$(4.3) \quad \sum_{k,l=1,2,3} \int_{\Omega} C_{kl}(x') \partial_{x_k} w(x', x_3) \partial_{x_l} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = 0.$$

For  $1 \leq k, l \leq 2$ , we notice that

$$\begin{aligned}(4.4) \quad \int_{\Omega} C_{kl}(x') \partial_{x_l} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{kl}(x') \partial_{x_k} \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' \\ &= \int_{\Omega} C_{kl}(x') \partial_{x_l} \widehat{w}(x', \xi) \partial_{x_k} (\varphi \otimes \psi)(x', \xi) dx' d\xi,\end{aligned}$$

directly from Lemma 4.1. Further, for all  $l = 1, 2$ , it holds true that

$$\begin{aligned}\int_{\Omega} C_{3l}(x') \partial_{x_l} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{3j}(x') \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{\psi}'(x_3) dx_3 \right) dx' \\ &= \int_{\omega} C_{3l}(x') \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_l} w(x', x_3) \widehat{(-i\xi)\psi}(x_3) dx_3 \right) dx' .\end{aligned}$$

Therefore we have

$$(4.5) \quad \int_{\Omega} C_{3l}(x') \partial_{x_l} w(x', x_3) \partial_{x_3} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{3l}(x') (-i\xi) \partial_{x_l} \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

Next, for  $k = 1, 2$ , we may write that

$$\begin{aligned}\int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\omega} C_{k3}(x') \partial_{x_k} \varphi(x') \left( \int_{\mathbb{R}} \partial_{x_3} w(x', x_3) \widehat{\psi}(x_3) dx_3 \right) dx' \\ &= - \int_{\omega} C_{k3}(x') \partial_{x_k} \varphi(x') \left( \int_{\mathbb{R}} w(x', x_3) \widehat{\psi}'(x_3) dx_3 \right) dx',\end{aligned}$$

with

$$\int_{\mathbb{R}} w(x', x_3) \widehat{\psi}'(x_3) dx_3 = - \int_{\mathbb{R}} w(x', x_3) \widehat{(i\xi)\psi}(x_3) dx_3 = - \int_{\mathbb{R}} (i\xi) \widehat{w}(x', \xi) \psi(\xi) d\xi,$$

hence

$$\int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{k3}(x') (i\xi) \widehat{w}(x', \xi) \partial_{x_k} (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

By integrating by parts in the right hand side of the above identity we obtain that

$$\begin{aligned}(4.6) \quad \int_{\Omega} C_{k3}(x') \partial_{x_3} w(x', x_3) \partial_{x_k} (\varphi \otimes \widehat{\psi})(x', x_3) dx' dx_3 &= \int_{\Omega} C_{k3}(x') (-i\xi) \partial_{x_k} \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi \\ &\quad + \int_{\Omega} \partial_{x_k} C_{k3}(x') (-i\xi) \widehat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.\end{aligned}$$

Finally, bearing in mind that  $\partial_{x_3} (\varphi \otimes \widehat{\psi}) = \varphi \otimes \widehat{(-i\xi)\psi}$  and noticing that

$$\int_{\mathbb{R}} \partial_{x_3} w(x', x_3) \widehat{(-i\xi)\psi}(x_3) dx_3 = - \int_{\mathbb{R}} w(x', x_3) \widehat{(-i\xi)^2 \psi}(x_3) dx_3 = - \int_{\mathbb{R}} \widehat{w}(x', \xi) (-i\xi)^2 \psi(\xi) d\xi,$$

we find out that

$$(4.7) \quad \int_{\Omega} C_{33}(x') \partial_{x_3} w(x', x_3) \partial_{x_3} (\varphi \otimes \hat{\psi})(x', x_3) dx' dx_3 = \int_{\Omega} C_{33}(x') (-i\xi)^2 \hat{w}(x', \xi) (\varphi \otimes \psi)(x', \xi) dx' d\xi.$$

Now, putting (4.3)-(4.7) together, we end up getting that

$$\langle -\operatorname{div}_{x'}(\tilde{C}(x') \nabla_{x'} \hat{w}) + P(x', \xi) \cdot \nabla_{x'} \hat{w} + q(x', \xi) \hat{w}, \Phi \rangle = 0, \quad \Phi \in C_0^\infty(\omega) \otimes C_0^\infty(\mathbb{R}),$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $C_0^\infty(\Omega)$  and  $\mathcal{D}'(\Omega)$ . From this and the density of  $C_0^\infty(\omega) \otimes C_0^\infty(\mathbb{R})$  in  $C_0^\infty(\Omega)$  then follows that

$$-\operatorname{div}_{x'}(\tilde{C}(x') \nabla_{x'} \hat{w}) + P(x', \xi) \cdot \nabla_{x'} \hat{w} + q(x', \xi) \hat{w} = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

which completes the proof.  $\square$

In the remaining of this section we assume that  $\theta(x_3) = ax_3 + b$ , where  $a$  and  $b$  are two fixed real numbers. With the help of Lemma 4.2 we may now re-express (1.3) in the Fourier plane. For the sake of simplicity, we shall write  $A_a(x')$  instead of  $A(x', \theta(x_3))$ , as we have  $\theta'(x_3) = a$  for all  $x_3 \in \mathbb{R}$ .

For  $g \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$  such that  $\int_{\mathbb{R}} g(x_3) dx_3 = 1$  and for  $h \in H^{1/2}(\partial\omega)$ , we consider the  $H^1(\Omega)$ -solution  $u$  to (1.3), with:

$$f(x', x_3) = g(x_3)h(x'), \quad x' \in \partial\omega, \quad x_3 \in \mathbb{R}.$$

Since  $u$  is solution to (4.1) with  $C = A_a \in W^{1,\infty}(\omega)^{3 \times 3}$ , we deduce from Lemma 4.2 that  $\hat{u} \in L^2(\mathbb{R}; H^1(\omega))$  is solution to the system

$$(4.8) \quad \begin{cases} -\operatorname{div}_{x'}(\tilde{A}_a(x') \nabla_{x'} \hat{u}(x', \xi)) - 2ia\xi x'^\perp \cdot \nabla_{x'} \hat{u} + \xi^2 \hat{u} = 0 & \text{in } \mathcal{D}'(\Omega), \\ \hat{u}(\cdot, \xi) = \hat{g}(\xi)f & \text{on } \partial\omega, \text{ for all } \xi \in \mathbb{R}, \end{cases}$$

where

$$x'^\perp = (-x_2, x_1) \text{ and } \tilde{A}_a(x') = \begin{pmatrix} 1 + x_2^2 a^2 & -x_2 x_1 a^2 \\ -x_2 x_1 a^2 & 1 + x_1^2 a^2 \end{pmatrix}.$$

We turn now to examining the variational problem associated to (4.8).

**Variational problem associated to (4.8).** Let us consider the bilinear form

$$\begin{aligned} \mathcal{A}_\xi[(v, w), (\varphi, \psi)] &= \int_{\omega} \tilde{A}_a \nabla v \cdot \nabla \varphi dx' - 2a\xi \int_{\omega} x'^\perp \cdot \nabla w \varphi dx' + \xi^2 \int_{\omega} v \varphi dx' \\ &\quad + \int_{\omega} \tilde{A}_a \nabla w \cdot \nabla \psi dx' + 2a\xi \int_{\omega} x'^\perp \cdot \nabla v \psi dx' + \xi^2 \int_{\omega} w \psi dx', \quad (v, w), (\varphi, \psi) \in \mathcal{H}, \end{aligned}$$

defined on the Hilbert space  $\mathcal{H} = H_0^1(\omega) \times H_0^1(\omega)$  endowed with the norm

$$\|(v, w)\|_{\mathcal{H}} = (\|\nabla v\|_{L^2(\omega)}^2 + \|\nabla w\|_{L^2(\omega)}^2)^{1/2}.$$

Taking into account that

$$\tilde{A}_a(x') \zeta \cdot \zeta \geq |\zeta|^2, \quad \text{for all } \zeta \in \mathbb{R}^2 \text{ and } x' \in \omega,$$

and that

$$2a|\xi| \int_{\omega} |x'^\perp \cdot \nabla v w| dx' \leq a^2 \delta^2 \int_{\omega} |\nabla v|^2 dx' + \xi^2 \int_{\omega} w^2 dx', \quad (v, w) \in \mathcal{H},$$

where  $\delta = \max_{x' \in \omega} |x'|$ , it is easy to see that

$$(4.9) \quad \mathcal{A}_\xi[(v, w), (v, w)] \geq (1 - a^2 \delta^2) \|(v, w)\|_{\mathcal{H}}^2.$$

Let us fix  $a_0 > 0$  so small that  $\alpha = 1 - a_0^2 \delta^2 > 0$ . Then, due to the above estimate, the bilinear form  $\mathcal{A}_\xi$  is  $\alpha$ -elliptic for every  $\xi \in \mathbb{R}$ , provided we have  $|a| \leq a_0$ . For each  $\Phi \in C(\mathbb{R}; \mathcal{H}')$  and every  $\xi \in \mathbb{R}$ , there is thus a unique  $(v(\xi), w(\xi)) \in \mathcal{H}$  satisfying

$$(4.10) \quad \mathcal{A}_\xi[(v(\xi), w(\xi)), (\varphi, \psi)] = \langle \Phi(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H},$$

by Lax-Milgram's lemma. From this then follows that

$$(4.11) \quad \begin{aligned} &\mathcal{A}_{\xi+\eta}[(v(\xi+\eta), w(\xi+\eta)) - (v(\xi), w(\xi)), (\varphi, \psi)] \\ &= \mathcal{A}_\xi[(v(\xi), w(\xi)), (\varphi, \psi)] - \mathcal{A}_{\xi+\eta}[(v(\xi), w(\xi)), (\varphi, \psi)] + \langle \Phi(\xi+\eta) - \Phi(\xi), (\varphi, \psi) \rangle, \end{aligned}$$

for each  $\xi, \eta \in \mathbb{R}$  and  $(\varphi, \psi) \in \mathcal{H}$ . Further, by noticing through elementary computations that

$$\mathcal{A}_{\xi+\eta}[(v, w), (\varphi, \psi)] = \mathcal{A}_{\xi+\eta}[(v, w), (\varphi, \psi)] - 2a\eta \int_{\omega} x'^{\perp} \cdot (\varphi \nabla_{x'} w - \psi \nabla_{x'} v) dx' + \eta(2\xi + \eta) \int_{\omega} (v\varphi + w\psi) dx',$$

for every  $(v, w), (\varphi, \psi) \in \mathcal{H}$ , we deduce from (4.11) and Poincaré's inequality that there is a constant  $C = C(\xi, \omega, a_0) > 0$  satisfying

$$\begin{aligned} & \mathcal{A}_{\xi}[(v(\xi), w(\xi)), (v(\xi + \eta) - v(\xi), w(\xi + \eta) - w(\xi))] \\ & - \mathcal{A}_{\xi+\eta}[(v(\xi), w(\xi)), (v(\xi + \eta) - v(\xi), w(\xi + \eta) - w(\xi))] \\ (4.12) \quad & \leq C|\eta| \|(v(\xi), w(\xi))\|_{\mathcal{H}} \|(v(\xi + \eta) - v(\xi), w(\xi + \eta) - w(\xi))\|_{\mathcal{H}}, \end{aligned}$$

for all  $\xi \in \mathbb{R}$  and  $\eta \in [-1, 1]$ . In light of (4.9), (4.11) written with  $(\varphi, \psi) = (v(\xi + \eta) - v(\xi), w(\xi + \eta) - w(\xi))$  and (4.12), we thus find out that

$$\alpha \|(v(\xi + \eta) - v(\xi), w(\xi + \eta) - w(\xi))\|_{\mathcal{H}} \leq C|\eta| \|(v(\xi), w(\xi))\|_{\mathcal{H}} + \|\Phi(\xi + \eta) - \Phi(\xi)\|_{\mathcal{H}'}.$$

This proves that  $(v, w) \in C(\mathbb{R}; \mathcal{H})$ . Moreover, we obtain

$$(4.13) \quad \|(v(\xi), w(\xi))\|_{\mathcal{H}} \leq (1/\alpha) \|\Phi(\xi)\|_{\mathcal{H}'}, \quad \xi \in \mathbb{R},$$

directly from (4.9)-(4.10). Further, it is easy to check for  $\Phi \in C^1(\mathbb{R}; \mathcal{H}')$  that  $(v'(\xi), w'(\xi)) \in C(\mathbb{R}, \mathcal{H})$  is the solution to the variational problem

$$\mathcal{A}_{\xi}[(v'(\xi), w'(\xi)), (\varphi, \psi)] = \langle \Phi_0(\xi), (\varphi, \psi) \rangle + \langle \Phi'(\xi), (\varphi, \psi) \rangle \quad \text{for all } (\varphi, \psi) \in \mathcal{H},$$

where we have set

$$\langle \Phi_0(\xi), (\varphi, \psi) \rangle = 2a \int_{\omega} x'^{\perp} \cdot (\varphi \nabla_{x'} w(\xi) - \psi \nabla_{x'} v(\xi)) dx' + 2\xi \int_{\omega} (v(\xi)\varphi + w(\xi)\psi) dx'.$$

Using (4.13) and putting  $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ , we deduce from the above estimate that

$$\|(v'(\xi), w'(\xi))\|_{\mathcal{H}} \leq C(\langle \xi \rangle \|\Phi(\xi)\|_{\mathcal{H}'} + \|\Phi'(\xi)\|_{\mathcal{H}'}), \quad \xi \in \mathbb{R},$$

for some constant  $C = C(a_0, \omega) > 0$ . Similarly, if  $\Phi \in C^2(\mathbb{R}; \mathcal{H}')$  then the same reasoning shows that  $(v, w) \in C^2(\mathbb{R}, \mathcal{H})$  and

$$\|(v''(\xi), w''(\xi))\|_{\mathcal{H}} \leq C(\langle \xi \rangle^2 \|\Phi(\xi)\|_{\mathcal{H}'} + \langle \xi \rangle \|\Phi'(\xi)\|_{\mathcal{H}'} + \|\Phi''(\xi)\|_{\mathcal{H}'}), \quad \xi \in \mathbb{R}.$$

Hence we have obtained the:

**Proposition 4.1.** *For every  $\Phi \in H^2(\mathbb{R}, \mathcal{H}')$  such that  $\langle \xi \rangle^{2-j} \Phi^{(j)} \in L^2(\mathbb{R}, \mathcal{H}')$ ,  $j = 0, 1$ , the variational problem (4.10) admits a unique solution  $(v, w) \in H^2(\mathbb{R}; \mathcal{H})$  satisfying*

$$(4.14) \quad \|(v, w)\|_{H^2(\mathbb{R}; \mathcal{H})} \leq C \left( \sum_{j=0}^2 \|\langle \xi \rangle^{2-j} \Phi^{(j)}\|_{L^2(\mathbb{R}; \mathcal{H}')} \right),$$

for some constant  $C = C(\omega, a_0) > 0$ . The above assumptions on  $\Phi$  are actually satisfied whenever  $\Phi = \widehat{\Psi}$  for some  $\Psi \in H^2(\mathbb{R}; \mathcal{H}')$  such that  $x_3 \Psi \in H^1(\mathbb{R}; \mathcal{H}')$  and  $x_3^2 \Psi \in L^2(\mathbb{R}; \mathcal{H}')$ . Moreover, the estimate (4.14) reads

$$(4.15) \quad \|(v, w)\|_{H^2(\mathbb{R}; \mathcal{H})} \leq C \left( \sum_{j=0}^2 \|x_3^j \Psi\|_{H^{2-j}(\mathbb{R}; \mathcal{H}')} \right),$$

in this case.

Armed with Proposition 4.1 we may now tackle the analysis of (4.8).

**Analysis of the solution to (4.8).** Pick  $F \in H^1(\omega)$  such that  $F = f$  on  $\partial\omega$  and  $\|F\|_{H^1(\omega)} \leq C(\omega) \|f\|_{H^{1/2}(\partial\omega)}$ . Let  $\tilde{u}^r$  (resp.  $\tilde{u}^i$ ) denote the real (resp. imaginary) part of  $\tilde{u} = \widehat{u} - \widehat{g}(\xi)F = \tilde{u}^r + i\tilde{u}^i$ . Since the Fourier transform  $\widehat{u}$  of the  $H^1(\Omega)$ -solution  $u$  to (1.3) is actually solution to (4.8), we get by direct calculation that  $(\tilde{u}^r, \tilde{u}^i)$  is solution to the variational problem (4.10), where

$$(4.16) \quad \langle \Phi(\xi), (\varphi, \psi) \rangle = -\mathcal{A}_{\xi}[(\widehat{g}^r F, \widehat{g}^i F), (\varphi, \psi)],$$

and  $\widehat{g}^r$  (resp.  $\widehat{g}^i$ ) stands for the real (resp. imaginary) part of  $\widehat{g}$ . Further, in light of (4.16) we check out using elementary computations that

$$(4.17) \quad \|\Phi(\xi)\|_{\mathcal{H}'} \leq C \langle \xi \rangle^2 |\widehat{g}(\xi)| \|f\|_{H^{1/2}(\partial\omega)}$$

$$(4.18) \quad \|\Phi'(\xi)\|_{\mathcal{H}'} \leq C (\langle \xi \rangle^2 |\widehat{g}'(\xi)| + \langle \xi \rangle |\widehat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)}$$

$$(4.19) \quad \|\Phi''(\xi)\|_{\mathcal{H}'} \leq C (\langle \xi \rangle^2 |\widehat{g}''(\xi)| + \langle \xi \rangle |\widehat{g}'(\xi)| + |\widehat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)},$$

for some constant  $C = C(a_0, \omega) > 0$ . Therefore we have  $\langle \xi \rangle^j \Phi^{(2-j)} \in L^2(\mathbb{R})$  for  $j = 0, 1, 2$ , provided  $\langle \xi \rangle^{4-j} \widehat{g}^j \in L^2(\mathbb{R})$ , this later condition being ensured by the assumption  $x_3^j g \in H^{4-j}(\mathbb{R})$ . From this and Proposition 4.1 then follows the:

**Corollary 4.1.** *Assume that  $g \in H^4(\mathbb{R})$  is such that  $x_3 g \in H^3(\mathbb{R})$ ,  $x_3^2 g \in H^2(\mathbb{R})$  and  $\int_{\mathbb{R}} g(x_3) dx_3 = 1$ . Then it holds true that  $\widehat{u} \in H^2(\mathbb{R}; H^1(\omega))$ . Moreover we have  $u \in L^1(\mathbb{R}; H^1(\omega))$  and  $U = \widehat{u}(\cdot, 0) = \int_{\mathbb{R}} u(\cdot, x_3) dx_3 \in H^1(\omega)$  is the variational solution to the following boundary value problem*

$$\begin{cases} \operatorname{div}_{x'}(\widetilde{A}_a \nabla_{x'} U) = 0 & \text{in } \omega \\ U = f & \text{on } \partial\omega. \end{cases}$$

In view of (4.8) and (4.10), we deduce from (4.17)-(4.19) that

$$\begin{aligned} \|\operatorname{div}_{x'}(\widetilde{A}_a \nabla_{x'} \widehat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C \langle \xi \rangle^4 |\widehat{g}(\xi)| \|f\|_{H^{1/2}(\partial\omega)} \\ \|\partial_{\xi} \operatorname{div}_{x'}(\widetilde{A}_a \nabla_{x'} \widehat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C (\langle \xi \rangle^5 |\widehat{g}'(\xi)| + \langle \xi \rangle^4 |\widehat{g}(\xi)|) \|f\|_{H^{1/2}(\partial\omega)} \\ \|\partial_{\xi}^2 \operatorname{div}_{x'}(\widetilde{A}_a \nabla_{x'} \widehat{u}(\cdot, \xi))\|_{L^2(\omega)} &\leq C (\langle \xi \rangle^6 |\widehat{g}(\xi)| + \langle \xi \rangle^5 |\widehat{g}'(\xi)| + \langle \xi \rangle^4 |\widehat{g}''(\xi)|) \|f\|_{H^{1/2}(\partial\omega)}, \end{aligned}$$

for some positive constant  $C$  depending only on  $a_0$  and  $\omega$ . From this and Corollary 4.1 follows the:

**Proposition 4.2.** *Let  $g \in H^6(\mathbb{R})$  be such that  $x_3 g \in H^5(\mathbb{R})$ ,  $x_3^2 g \in H^4(\mathbb{R})$  and  $\int_{\mathbb{R}} g(x_3) dx_3 = 1$ . Then we have  $\widetilde{A}_a \nabla_{x'} \widehat{u} \cdot \nu(x') \in H^2(\mathbb{R}; H^{-1/2}(\partial\omega))$  and thus  $A_a \nabla u \cdot \nu(x) \in L^1(\mathbb{R}; H^{-1/2}(\partial\omega))$ , with*

$$\widetilde{A}_a \nabla_{x'} U \cdot \nu(x') = \widetilde{A}_a \nabla_{x'} \widehat{u}(\cdot, 0) \cdot \nu(x') = \int_{\mathbb{R}} A_a \nabla u(\cdot, x_3) \cdot \nu(x') dx_3 \in H^{-1/2}(\partial\omega).$$

In light of Proposition 4.2 we now introduce the two following DN maps:

$$\begin{aligned} \Lambda_a : f \in H^{1/2}(\partial\omega) &\rightarrow A_a \nabla u(\cdot, x_3) \cdot \nu(x) \in L^1(\mathbb{R}; H^{-1/2}(\partial\omega)) \\ \widetilde{\Lambda}_a : f \in H^{1/2}(\partial\omega) &\rightarrow \widetilde{A}_a \nabla_{x'} U \cdot \nu(x') \in H^{-1/2}(\partial\omega). \end{aligned}$$

These two operators are bounded, and they satisfy the estimate

$$(4.20) \quad \|\widetilde{\Lambda}_1 - \widetilde{\Lambda}_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), H^{-1/2}(\partial\omega))} \leq \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), L^1(\mathbb{R}; H^{-1/2}(\partial\omega)))},$$

where, for the sake of simplicity, we write  $\Lambda_j$  (resp.  $\widetilde{\Lambda}_j$ ) for  $\Lambda_{a_j}$  (resp.  $\widetilde{\Lambda}_{a_j}$ ),  $j = 1, 2$ .

Finally, the last step in the analysis of the case of affine twisting functions involves noticing by direct calculation that the matrix  $\partial_a \widetilde{A}(x', a)$  has two eigenvalues  $\lambda_0 = 0$  and  $\lambda_1 = |x'|^2$ . Hence, by mimicking the proof of [AG1][CLAIM, page 169], we obtain the:

**Theorem 4.1.** *Let  $a_0 > 0$  and let  $g$  be the same as in Proposition 4.2. Assume moreover that  $1 - a_0^2 \delta^2 > 0$ . Then there exists a constant  $C > 0$ , depending only on  $a_0$  and  $\omega$ , such that the following stability estimate*

$$|a_1 - a_2| \leq C \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(H^{1/2}(\partial\omega), L^1(\mathbb{R}; H^{-1/2}(\partial\omega)))},$$

holds true whenever

$$|a_1|, |a_2| \leq a_0.$$



## 5. THE DN MAP FOR THE ORIGINAL PROBLEM

We first start by defining the trace space for functions in  $H^1(\Omega_\theta)$ . To this purpose we set for all  $L > 0$ ,

$$\Omega_\theta^L = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \omega, x_3 \in (-L, L)\}$$

and

$$\Gamma_\theta^L = \{(R_{\theta(x_3)}x', x_3); x' = (x_1, x_2) \in \partial\omega, x_3 \in [-L, L]\}.$$

Given  $L > 0$ , it holds true for every  $u \in H^1(\Omega_\theta)$  that  $u \in H^1(\Omega_\theta^L)$ , whence  $u|_{\partial\Omega_\theta^L} \in H^{1/2}(\partial\Omega_\theta^L)$ . Putting

$$H^{1/2}(\Gamma_\theta^L) = \{h = g|_{\Gamma_\theta^L} \text{ in } L^2(\Gamma_\theta^L); g \in H^{1/2}(\partial\Omega_\theta^L)\},$$

we thus have  $u|_{\Gamma_\theta^L} \in H^{1/2}(\Gamma_\theta^L)$ . Here  $H^{1/2}(\Gamma_\theta^L)$  is equipped with its natural (quotient) norm

$$\|h\|_{H^{1/2}(\Gamma_\theta^L)} = \inf\{\|g\|_{H^{1/2}(\partial\Omega_\theta^L)}; g|_{\Gamma_\theta^L} = h\}.$$

Further we define

$$H_{\text{loc}}^{1/2}(\partial\Omega_\theta) = \{h \in L_{\text{loc}}^2(\partial\Omega_\theta); h|_{\Gamma_\theta^L} \in H^{1/2}(\Gamma_\theta^L) \text{ for all } L > 0\},$$

and introduce the following subspace of  $H_{\text{loc}}^{1/2}(\partial\Omega_\theta)$ :

$$\tilde{H}^{1/2}(\partial\Omega_\theta) = \{h \in H_{\text{loc}}^{1/2}(\partial\Omega_\theta); \text{ there exists } v \in H^1(\Omega_\theta) \text{ such that } v|_{\partial\Omega_\theta} = h\}.$$

Here and henceforth  $v|_{\partial\Omega_\theta} = h$  means that  $v|_{\Gamma_\theta^L} = h|_{\Gamma_\theta^L}$  in the trace sense for all  $L > 0$ . It can be checked that  $\tilde{H}^{1/2}(\partial\Omega_\theta)$  is a Banach space for the quotient norm

$$\|h\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} = \inf\{\|v\|_{H^1(\Omega_\theta)}; v|_{\partial\Omega_\theta} = h\}.$$

Let us now introduce the mapping

$$\begin{aligned} I_\theta : C_0^1(\partial\Omega_\theta) &\longrightarrow C_0^1(\partial\Omega) \\ g &\longrightarrow f = g \circ \varphi_\theta, \end{aligned}$$

where, for the sake of shortness, we note

$$\varphi_\theta(x) = T_{\theta(x_3)}(x', x_3).$$

Further, pick  $g$  in  $C_0^1(\partial\Omega_\theta)$  and let  $v \in C_0^1(\mathbb{R}^3)$  be such that  $v|_{\partial\Omega_\theta} = g$ . Setting  $u = v|_{\Omega_\theta} \circ \varphi_\theta$ , we notice that

$$\|I_\theta g\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \leq C(\omega)\|u\|_{H^1(\Omega)} \leq C(\omega, \theta)\|v\|_{H^1(\Omega_\theta)}.$$

Therefore, we have

$$(5.1) \quad \|I_\theta g\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \leq C(\omega, \theta)\|g\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} \quad \text{for any } g \in C_0^1(\partial\Omega_\theta).$$

Let us now consider  $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$  and  $v \in H^1(\Omega_\theta)$  such that  $v|_{\partial\Omega_\theta} = g$ . For any sequence  $(v_n)_n \in C_0^1(\mathbb{R}^3)$  such that  $v_n|_{\Omega_\theta} \longrightarrow v$  in  $H^1(\Omega_\theta)$ , it is clear that

$$\|g - g_n\|_{\tilde{H}^{1/2}(\partial\Omega_\theta)} \leq \|v - v_n\|_{H^1(\Omega_\theta)},$$

provided we have  $g_n = v_n|_{\partial\Omega_\theta}$ . Hence  $(g_n)_n$  converges to  $g$  in  $\tilde{H}^{1/2}(\partial\Omega_\theta)$

For all  $n \geq 1$ , put  $f_n = I_\theta g_n = g_n \circ \varphi_\theta$  and  $u_n = v_n \circ \varphi_\theta$ . Since  $f_n = u_n|_{\partial\Omega}$ , we see that

$$\|f_n - f_m\|_{H^1(\mathbb{R}; H^{1/2}(\partial\omega))} \leq C(\omega)\|u_n - u_m\|_{H^1(\Omega)} \leq C(\omega, \theta)\|v_n - v_m\|_{H^1(\Omega_\theta)}.$$

Consequently,  $(f_n)_n$  is a Cauchy sequence in  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  so there exists  $f = \lim_n f_n \in H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ . Set  $f = I_\theta g$ . In view of (5.1),  $I_\theta$  extends to a bounded operator, still denoted by  $I_\theta$ , from  $\tilde{H}^{1/2}(\partial\Omega_\theta)$  into  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$ .

Arguing as above, we find out that the mapping

$$\begin{aligned} J_\theta : C_0^1(\partial\Omega) &\longrightarrow C_0^1(\partial\Omega_\theta) \\ f &\longrightarrow g = f \circ \psi_\theta, \end{aligned}$$

where  $\psi_\theta = \varphi_\theta^{-1}$ , extends to a bounded operator, which is still called  $J_\theta$ , from  $H^1(\mathbb{R}; H^{1/2}(\partial\omega))$  into  $\tilde{H}^{1/2}(\partial\Omega_\theta)$ .

Evidently, we have  $I_\theta J_\theta f = f$  for all  $f \in C_0^1(\partial\Omega)$  and  $J_\theta I_\theta g = g$  for all  $g \in C_0^1(\partial\Omega_\theta)$ . Therefore  $J_\theta = I_\theta^{-1}$ , by density.

Next, by reasoning in the same way as in the derivation of (1.3), we prove with the help of the Lax-Milgram's lemma that the boundary value problem (1.1) has a unique solution  $v \in H^1(\Omega_\theta)$  for every  $g \in \tilde{H}^{1/2}(\partial\Omega_\theta)$ . Moreover the operator  $\tilde{\Lambda}_\theta$  is well defined as a bounded operator from  $\tilde{H}^{1/2}(\partial\Omega_\theta)$  into its dual space  $\tilde{H}^{-1/2}(\partial\Omega_\theta)$ . Similarly to  $\Lambda_\theta$ , it can be checked that  $\tilde{\Lambda}_\theta$  is characterized by the identity

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \int_{\Omega_\theta} \nabla v \cdot \nabla H dy,$$

which holds true for all  $h \in \tilde{H}^{1/2}(\partial\Omega_\theta)$  and  $H \in H^1(\Omega_\theta)$  such that  $H|_{\partial\Omega_\theta} = h$ . By performing the change of variable  $y = \varphi_\theta(x)$  in the last integral, we get that

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \int_{\Omega} A \nabla u \cdot \nabla (H \circ \varphi_\theta) dx,$$

where  $u$  is the solution of the boundary value problem (1.3) with  $f = I_\theta g$ . Therefore we have

$$\langle \tilde{\Lambda}_\theta g, h \rangle = \langle \Lambda_\theta I_\theta g, I_\theta h \rangle,$$

which means that  $\tilde{\Lambda}_\theta = I_\theta^* \Lambda_\theta I_\theta$ , or equivalently that  $\Lambda_\theta = J_\theta^* \tilde{\Lambda}_\theta J_\theta$ .

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